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AUTHOR(S):

ISHIKAWA, Masao

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A MIXED KNUTH CORRESPONDENCE FOR (A, B) -PARTIALLY STRICT TABLEAUX

Masao ISHIKAWA

Department of Mathematics

Faculty of Science

University of Tokyo

Hongo, Bunkyo-ku Tokyo, Japan 113

Fix finite totally ordered sets \mathcal{A} , \mathcal{A}' throughout this article. A pair (U, C) of subsets of \mathcal{A} is called a *division* of \mathcal{A} if it satisfies

$$U \uplus C = \mathcal{A}. \quad (\text{disjoint union})$$

Henceforth, we fix a division (U, C) of \mathcal{A} , and we call elements of U *uncircled letters* and elements of C *circled letters*. Fix another division (A, B) of \mathcal{A} . Set $k = |A|$ and $l = |B|$ so that we have $|\mathcal{A}| = k + l$. We have two pairs (A, B) and (U, C) which are divisions of \mathcal{A} . We write

$$\begin{aligned} A_u &= A \cap U, & A_c &= A \cap C \\ B_u &= B \cap U, & B_c &= B \cap C \end{aligned}$$

EXAMPLE 1.1

Set $A = \{1, 3^\circ, 5, 7^\circ\}$, $B = \{2, 4^\circ, 6, 8^\circ\}$, $U = \{1, 2, 5, 6\}$ and $C = \{3^\circ, 4^\circ, 7^\circ, 8^\circ\}$. Then (A, B) and (U, C) are divisions of $[8]$ and we have $A_u = \{1, 5\}$, $A_c = \{3^\circ, 7^\circ\}$, $B_u = \{2, 6\}$, and $B_c = \{4^\circ, 8^\circ\}$. As in this example we write elements of A in lightface and elements of B in boldface.

We take the word “ (A, B) -partially strict” from [Ok], but the original definition is due to [St]. For the definition of (k, l) -semistandard tableaux see [BR] or [Re]. A *reverse plane partition* π is a filling of a Young diagram with letters of \mathcal{A} wherein in each row from left to right and in each column from top to bottom the letters are arranged in weakly increasing order.

DEFINITION 1.1

Let π be a reverse plane partition. π is said to be (A, B) -partially strict if it satisfies the conditions:

- (i) For any $m \in A$, m appears at most once in each column.
- (ii) For any $m \in B$, m appears at most once in each row.

We call a (A, B) -partially strict reverse plane partition a (A, B) -partially strict tableau. A (\mathbf{P}, \emptyset) -partially strict skew tableau is usually called a column-strict skew tableau and a (\emptyset, \mathbf{P}) -partially strict skew tableau, a row-strict skew tableau. If $A = \{1, 2, \dots, k\}$ and $B = \{1', 2', \dots, l'\}$, where $1 < 2 < \dots < k < 1' < 2' < \dots < l'$, then a (A, B) -partially strict tableau is called a (k, l) -semistandard tableau.

EXAMPLE 1.2

Set (A, B) to be the division given in EXAMPLE 4.1. Then

$$\pi =$$

1	1	1	2	$^\circ 4$	5	5	$^\circ 7$
$^\circ 3$	$^\circ 3$	$^\circ 3$	$^\circ 4$	6	$^\circ 7$	$^\circ 7$	
$^\circ 4$	5	5	6				
$^\circ 4$	$^\circ 7$	$^\circ 7$	$^\circ 8$				
5	$^\circ 8$						

is a (A, B) -partially strict tableau.

$$\pi' = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 3 & 2' & 3' \\ \hline 4 & 4 & 1' & 2' & \\ \hline 1' & 3' & & & \\ \hline 1' & & & & \\ \hline \end{array}$$

is an exmple of $(4, 3)$ -semistandard tableau, where $1 < 2 < 3 < 4 < 1' < 2' < 3'$.

DEFINITION 1.2

Let λ/μ be a skew diagram. Let $\mathcal{T}_{(A,B)}(\lambda/\mu)$ denote the set of all (A, B) -partially strict skew tableaux of shape λ/μ . For $\pi \in \mathcal{T}_{(A,B)}(\lambda/\mu)$ set the weight $wt(\pi)$ of π to be $\prod_{a \in A} x_a^{m_a}$ where

$$m_a = \text{the number of times } a \text{ occurs in } \pi$$

and x_a 's are indeterminates. Set

$$HS_{\lambda/\mu}^{(A,B)}(x) = \sum_{\pi \in \mathcal{T}_{(A,B)}(\lambda/\mu)} wt(\pi).$$

It is clear from the defintion that

$$HS_{\lambda/\mu}^{(A,B)}(x) = HS_{\lambda'/\mu'}^{(B,A)}(x).$$

In particular if $A = \{1, 2, \dots, k\}$ and $B = \{1', 2', \dots, l'\}$, where $1 < 2 < \dots < k < 1' < 2' < \dots < l'$, we write $HS_{\lambda/\mu}^{(A,B)}(x)$ as $HS_{\lambda/\mu}^{(k,l)}(x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_l)$.

PROPOSITION 1.1

Set $A = \{a_1, a_2, \dots, a_k\}$ and $B = \{b_1, b_2, \dots, b_l\}$. Then

$$HS_{\lambda/\mu}^{(A,B)}(x) = HS_{\lambda/\mu}^{(k,l)}(x_{a_1}, x_{a_2}, \dots, x_{a_k}, x_{b_1}, x_{b_2}, \dots, x_{b_l})$$

Proof.

We can easily construct a bijection between $\mathcal{T}_{(A,B)}(\lambda/\mu)$ and $\mathcal{T}_{(\{1,2,\dots,k\},\{1',2',\dots,l'\})}(\lambda/\mu)$ using the jeu de taquin method in [Re], Section 3, pp.266. For details see [Re]. ■

Now we define a mixed Knuth insertion.

DEFINITION 1.3

Let π be a (A, B) -partially strict tableau and let $x \in A$. We define $\text{INSERT}_{(A,B;U,C)}(x)$ as follow.

If $x \in U$, insert x into the first row of π ; if $x \in C$, insert x into the first column of π . If the bumped element y is uncircled, then we insert y into the row immediately below or if the bumped element y is circled, then we insert y into the column immediately to its right by the following rules.

y replace the least element which is $> y$ if $y \in A_u \cup B_c$: or y replace the least element which is $\geq y$ if $y \in B_u \cup A_c$.

Continue until an insertion takes place at the end of a row or column, bumping no new element. This procedure terminates in a finite number of steps. Then set (s, t) to be the cell which is added to π .

Similarly we define $\overline{\text{INSERT}}_{(A,B;U,C)}(x)$ by swapping U and C in the foregoing definition. If $x \in U$, insert x into the first column of π ; if $x \in C$, insert x into the first row of π . The uncircled letters which are bumped are inserted into the column immediately to its right and circled letters are inserted into the row immediately below by the following rule.

y replace the least element which is $> y$ if $y \in A_c \cup B_u$: or y replace the least element which is $\geq y$ if $y \in B_c \cup A_u$.

It is easy to see that the resulting tableau is also (A, B) -partially strict. Let $\pi \leftarrow^m x$ (resp. $x \rightarrow^m \pi$) denote the tableau which is obtained after we applied $\text{INSERT}_{(A,B;U,C)}(x)$ (resp. $\overline{\text{INSERT}}_{(A,B;U,C)}(x)$) to π .

EXAMPLE 1.3

Let π be the (A, B) -partially strict tableau in EXAMPLE 4.2.

$$\pi \leftarrow^m 4 =$$

1	1	1	1	°4	5	5	°7
2	°3	°3	°3	°4	°7	°7	
°4	5	5	6	°8			
°4	6	°7	°7				
5	°8						

And we have $(s, t) = (4, 4)$.

$$2 \rightarrow^m \pi =$$

1	1	1	2	°4	5	5	°7
2	°3	°3	°4	6	°7	°7	
°3	5	5	5	6			
°4	°7	°7	°8				
°4	°8						

And we have $(s, t) = (6, 1)$.

REMARK 1.1

In [Re] two insertion procedures are defined for (k, l) -semistandard tableaux. Set $A = \{1, 2, \dots, k\}$ and $B = \{1', 2', \dots, l'\}$, where $1 < 2 < \dots < k < 1' < 2' < \dots < l'$. If $U = A$ and $C = B$, then the insertion algorithm in Definition 4.2 is called RS1 insertion in [Re]. If $U = \mathcal{A}$ and $C = \emptyset$, then the insertion algorithm is called RS2 insertion.

DEFINITION 1.4

Let π be a (A, B) -partially strict tableau. Set m_x to be the number of times x occurs in π for each $x \in \mathcal{A}$. Let $m = \sum_{x \in \mathcal{A}} m_x$. We make a partial tableau $pt(\pi)$ with letters in $[m]$ from π as follows. If $x \in A$, then replace m_x x 's in π to $\sum_{y < x} m_x + 1, \sum_{y < x} m_x + 2, \dots, \sum_{y \leq x} m_x$ from left to right. If $x \in B$, then replace m_x x 's in π to $\sum_{y < x} m_x + 1, \sum_{y < x} m_x + 2, \dots, \sum_{y \leq x} m_x$ from top to bottom. If $x \in U$ then $\sum_{y < x} m_x + 1, \sum_{y < x} m_x + 2, \dots, \sum_{y \leq x} m_x$ are in U , and vice versa.

EXAMPLE 1.4

If π is as in Example 4.2, then $pt(\pi)$ is as in Example 2.2.

DEFINITION 1.5

A *word with repetition* is a sequence $w = w_1 w_2 \dots w_m$ of letters in \mathcal{A} wherein each $a \in \mathcal{A}$ can appear more than once. Given a word with repetition $w = w_1 w_2 \dots w_m$, we make the insertion tableau $\pi = \emptyset \leftarrow^m w$ for w as follows. For $i = 1, 2, \dots, m$ we define inductively $\pi_0 = \emptyset$ and $\pi_i = \pi_{i-1} \leftarrow^m w_i$. Let $\pi = \pi_m$.

EXAMPLE 1.5

$$w = {}^\circ 2 \quad {}^\circ 2 \quad 1 \quad {}^\circ 3 \quad 4 \quad {}^\circ 3 \quad {}^\circ 3 \quad 1 \quad 4 \quad {}^\circ 2 \quad 4$$

is a word with repetition and the insertion tableau for w is as follows.

$$\emptyset \leftarrow^m w = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & {}^\circ 2 & {}^\circ 3 & 4 \\ \hline {}^\circ 2 & {}^\circ 3 & {}^\circ 3 & 4 & \\ \hline {}^\circ 2 & 4 & & & \\ \hline \end{array}$$

DEFINITION 1.6

For a given word with repetition $w = w_1 w_2 \dots w_m$ we make a permutation $p(w)$ of $[m]$ as follows. For each $x \in \mathcal{A}$ let m_x denote the number of times x appears in w . For each $x \in \mathcal{A}$, if $x \in A_u \cup B_c$ then replace all x in w by $\sum_{y < x} m_x + 1, \sum_{y < x} m_x + 2, \dots, \sum_{y \leq x} m_x$ in increasing order. For each $x \in \mathcal{A}$, if $x \in A_c \cup B_u$ then replace all x in w by $\sum_{y \leq x} m_x, \sum_{y \leq x} m_x - 1, \dots, \sum_{y < x} m_x + 1$ in decreasing order. If $x \in U$ then $\sum_{y < x} m_x + 1, \sum_{y < x} m_x + 2, \dots, \sum_{y \leq x} m_x$ are in U , and vice versa.

EXAMPLE 1.6

Let w be as in Example 4.5.

$$p(w) = {}^\circ 3 \quad {}^\circ 4 \quad 1 \quad {}^\circ 8 \quad 9 \quad {}^\circ 7 \quad {}^\circ 6 \quad 2 \quad 10 \quad {}^\circ 5 \quad 11$$

$$\emptyset \leftarrow^m p(w) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & {}^\circ 8 & {}^\circ 3 & 9 \\ \hline {}^\circ 4 & {}^\circ 6 & {}^\circ 7 & 10 & \\ \hline {}^\circ 5 & 11 & & & \\ \hline \end{array}$$

The following proposition is easy to see from definitions.

PROPOSITION 1.2

Let w be a word with repetition. Let π be the insertion tableau of w . Then the following diagram commutes.

$$\begin{array}{ccc} w & \longleftrightarrow & \pi \\ \downarrow p & & \downarrow p^t \\ p(w) & \longleftrightarrow & p^t(\pi) \end{array}$$

where the top and bottom bijections are the mixed Knuth and mixed Robinson-Schensted maps, respectively.

■

LEMMA 1.1

Let π be a (A, B) -partially strict tableau and $x, x' \in \mathcal{A}$. If $\text{INSERT}_{(A, B; U, C)}(x)$, determining s and t , is immediately followed by $\text{INSERT}_{(A, B; U, C)}(x')$, determining (s', t') , then

(Case 1) $x, x' \in U$

- (a) If $x < x'$ or $x = x' \in A$ then we have $s \geq s'$ and $t < t'$.
- (b) If $x > x'$ or $x = x' \in B$ then we have $s < s'$ and $t \geq t'$.

(Case 2) $x, x' \in C$

- (a) If $x > x'$ or $x = x' \in A$ then $s \geq s'$ and $t < t'$.
- (b) If $x < x'$ or $x = x' \in B$ then $s < s'$ and $t \geq t'$.

(Case 3) $x \in U$ and $x' \in C$, we always have $s < s'$ and $t \geq t'$.

(Case 4) $x \in C$ and $x' \in U$, we always have $s \geq s'$ and $t < t'$.

Proof.

Choose arbitrary word with repetition w such that $\pi = \emptyset \leftarrow^m w$. Let $w' = wxx'$. Then it is easy to verify the lemma by using Proposition 4.2, Corollary 2.2, and Lemma 1.1. For example, we verify Case 3. Assume that $x \in U$ and $x' \in C$. Then $x' \in C$ is changed into some negative letter $-x'$ which is less than x so that we obtain $s < s'$ and $t \geq t'$ immediately by Lemma 1.1. ■

REMARK 1.2

In the foregoing lemma by changing $\text{INSERT}_{(A,B;U,C)}(x)$ and $\text{INSERT}_{(A,B;U,C)}(x')$ into $\overline{\text{INSERT}}_{(A,B;U,C)}(x)$ and $\overline{\text{INSERT}}_{(A,B;U,C)}(x')$, respectively and swapping U and C , we obtain a similar result on $\overline{\text{INSERT}}_{(A,B;U,C)}(\cdot)$.

Fix another finite totally ordered set \mathcal{A}' and its divisions (A', B') and (U', C') such that $|A'| = k'$ and $|B'| = l'$. We write

$$\begin{aligned} A'_u &= A' \cap U', & A'_c &= A' \cap C' \\ B'_u &= B' \cap U', & B'_c &= B' \cap C'. \end{aligned}$$

DEFINITION 1.7

Let a be a $(k' + l') \times (k + l)$ matrix of nonnegative integers

$$a = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

whose rows are labeled by elements of \mathcal{A}' and columns are labeled by elements of \mathcal{A} . a is said to be *admissible* if it satisfies:

- (1) If $(i, j) \in A' \times A \cup B' \times B$, $a_{i,j} \in \mathbf{N}$.
- (2) If $(i, j) \in A' \times B \cup B' \times A$, $a_{i,j} \in \{0, 1\}$.

Let $\mathcal{M}(A', B', A, B)$ denote the set of all admissible $(k' + l') \times (k + l)$ matrices.

EXAMPLE 1.7

Let $A' = \{2, {}^\circ 4\}$, $B' = \{1, {}^\circ 3\}$, $A = \{{}^\circ 3, 4\}$, and $B = \{1, {}^\circ 2\}$. Then

$$a = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{pmatrix}$$

is a admissible 4×4 matrix. As in this example we write $a_{i,j}$ such that $(i, j) \in A' \times A \cup B' \times B$ in italic.

DEFINITION 1.8

Let $a \in \mathcal{M}(A', B', A, B)$. From a we make a two-line array

$$l(a) = \begin{pmatrix} u_1 & u_2 & \cdots & \cdots & u_m \\ v_1 & v_2 & \cdots & \cdots & v_m \end{pmatrix}$$

as follows. We arrange $a_{u,v}$ pairs of row and column labels $\begin{pmatrix} u \\ v \end{pmatrix}$ by the following rule.

First we assume that

$$u_1 \leq u_2 \leq \cdots \leq u_m.$$

- (1) For each $u \in A'_u \cup B'_c$ we arrange all labels $\begin{pmatrix} u_i \\ v_i \end{pmatrix}$ such that $u_i = u$ as follows.

$$\underbrace{v_{p_1}, v_{p_2}, v_{p_3}, \dots, v_{p_r}}_{\substack{\text{elements of } C \\ \text{in decreasing order}}} \underbrace{v_{p_{r+1}}, v_{p_{r+2}}, \dots, v_{p_{r+s}}}_{\substack{\text{elements of } U \\ \text{in increasing order}}}$$

- (2) For each $u \in A'_c \cup B'_u$ we arrange all labels $\begin{pmatrix} u_i \\ v_i \end{pmatrix}$ such that $u_i = u$ as follows.

$$\underbrace{v_{p_1}, v_{p_2}, v_{p_3}, \dots, v_{p_r}}_{\substack{\text{elements of } U \\ \text{in decreasing order}}} \underbrace{v_{p_{r+1}}, v_{p_{r+2}}, \dots, v_{p_{r+s}}}_{\substack{\text{elements of } C \\ \text{in increasing order}}}$$

It is easy to see this gives an one to one correspondence between admissible matrices and two line arrays satisfying the above conditions. We call this two line array the *matrix word* of a and denote by $l(a)$. The top (resp. bottom) line of $l(a)$ is denoted by $\hat{l}(a) = u_1, u_2, \dots, u_m$ (resp. $\check{l}(a) = v_1, v_2, \dots, v_m$).

EXAMPLE 1.8

The two line array which correspond to the matrix a in EXAMPLE 4.5 is

$$l(a) = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 2 & \circ 3 & \circ 3 & \circ 3 & \circ 3 & \circ 4 & \circ 4 & \circ 4 & \circ 4 \\ 1 & 1 & \circ 3 & \circ 2 & 4 & 4 & \circ 3 & \circ 2 & \circ 2 & \circ 2 & 4 & 4 & 1 & \circ 3 \end{pmatrix}.$$

DEFINITION 1.9

Let $a \in \mathcal{M}(A', B', A, B)$. From a we make a two-line array $l(a)$ in Definition 4.8.

$$l = \begin{pmatrix} u_1 & u_2 & \cdots & \cdots & u_m \\ v_1 & v_2 & \cdots & \cdots & v_m \end{pmatrix}$$

We construct a sequence of tableaux pairs:

$$(\emptyset, \emptyset) = (\pi_0, \sigma_0), (\pi_1, \sigma_1), \dots, (\pi_m, \sigma_m) = (\pi, \sigma)$$

inductively as follows. For each $i = 1, 2, \dots, m$ form π_i from π_{i-1} by performing $\text{INSERT}_{(A, B; U, C)}(v_i)$ on π_{i-1} if u_i is a uncircled letter, or performing $\overline{\text{INSERT}}_{(A, B; U, C)}(v_i)$ on π_{i-1} if u_i is a circled letter. Form σ_i from σ_{i-1} by placing u_i on σ_{i-1} in the cell added to π_i . By Lemma 4.1 σ is a (A', B') -partially strict tableau and π and σ have the same shape.

EXAMPLE 1.9

Let a be as in Example 4.7. Then

$$\pi = \begin{array}{|c|c|c|c|c|c|} \hline 1 & \circ 2 & \circ 3 & \circ 3 & 4 & 4 \\ \hline 1 & \circ 2 & 4 & & & \\ \hline 1 & \circ 3 & & & & \\ \hline \circ 2 & 4 & & & & \\ \hline \circ 2 & & & & & \\ \hline \end{array} \quad \sigma = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 2 & 2 & \circ 3 & \circ 4 \\ \hline 1 & \circ 3 & \circ 4 & & & \\ \hline 2 & \circ 3 & & & & \\ \hline \circ 3 & \circ 4 & & & & \\ \hline \circ 4 & & & & & \\ \hline \end{array}$$

DEFINITION 1.10

Let $a \in \mathcal{M}(A', B', A, B)$. Let

$$l(a) = \begin{pmatrix} u_1 & u_2 & \cdots & \cdots & u_m \\ v_1 & v_2 & \cdots & \cdots & v_m \end{pmatrix}$$

be the two line array which correspond to a . We construct a biword w from l as follow. For each $x \in \mathcal{A}$ (resp. $x \in \mathcal{A}'$) let m_x (resp. m'_x) denote the number of times x occurs in the bottom (resp. top) line of l . Replace m'_x x 's in the top line of l by $\sum_{y < x} m_x + 1, \sum_{y < x} m_x + 2, \dots, \sum_{y \leq x} m_x$ from left to right. The circles are transferred unchanged in this replacement. For each $x \in \mathcal{A}$ let r_x (resp. s_x) be the number of pairs $\begin{pmatrix} u_i \\ v_j \end{pmatrix}$ such that $v_j = x$ and $u_i \in U$ (resp. $u_i \in C$). So we have $r_x + s_x = m_x$. For each $x \in \mathcal{A}$ we replace v_j 's such that $v_j = x$ by the following rules. The circles are transferred unchanged in this replacement.

(Case 1) : $x \in A_u \cup B_c$

Replace the v_j 's of pairs $\begin{pmatrix} u_i \\ v_j \end{pmatrix}$ such that $v_j = x$ and $u_i \in C$ by $\sum_{y < x} m_x + 1, \sum_{y < x} m_x + 2, \dots, \sum_{y < x} m_x + s_x$ from right to left. Then replace the v_j 's of pairs $\begin{pmatrix} u_i \\ v_j \end{pmatrix}$ such that $v_j = x$ and $u_i \in U$ by $\sum_{y < x} m_x + s_x + 1, \sum_{y < x} m_x + s_x + 2, \dots, \sum_{y \leq x} m_x$ from left to right.

(Case 2) : $x \in A_c \cup B_u$

Replace the v_j 's of pairs $\begin{pmatrix} u_i \\ v_j \end{pmatrix}$ such that $v_j = x$ and $u_i \in U$ by $\sum_{y < x} m_x + 1, \sum_{y < x} m_x + 2, \dots, \sum_{y < x} m_x + r_x$ from right to left. Then replace the v_j 's of pairs $\begin{pmatrix} u_i \\ v_j \end{pmatrix}$ such that $v_j = x$ and $u_i \in C$ by $\sum_{y < x} m_x + r_x + 1, \sum_{y < x} m_x + r_x + 2, \dots, \sum_{y \leq x} m_x$ from left to right.

Let $p(a)$ denote the resulting biword.

EXAMPLE 1.10

Let a be as Exemple 4.7 and $l(a)$ as Example 4.8. Then we have

$$p(a) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \circ 7 & \circ 8 & \circ 9 & \circ 10 & \circ 11 & \circ 12 & \circ 13 & \circ 14 \\ 2 & 1 & \circ 8 & \circ 7 & 13 & 14 & \circ 9 & \circ 6 & \circ 5 & \circ 4 & 12 & 11 & 3 & \circ 10 \end{pmatrix}.$$

The following proposition is easy to see from definitions.

PROPOSITION 1.3

Let $a \in \mathcal{M}(A', B', A, B)$. Then the following diagram commutes.

$$\begin{array}{ccc} a & \longleftrightarrow & (\pi, \sigma) \\ \downarrow p & & \downarrow \\ p(a) & \longleftrightarrow & (pt(\pi), pt(\sigma)) \end{array}$$

where the top and bottom bijections are the mixed Knuth and mixed Robinson-Schensted maps, respectively.

EXAMPLE 1.11

Let $p(a)$ be as in Example 4.9. Then the insertion pair of $p(a)$ is as follows.

$$\pi = \begin{array}{|c|c|c|c|c|c|} \hline 1 & \circ 4 & \circ 9 & \circ 10 & 13 & 14 \\ \hline 2 & \circ 5 & 12 & & & \\ \hline 3 & \circ 8 & & & & \\ \hline \circ 6 & 11 & & & & \\ \hline \circ 7 & & & & & \\ \hline \end{array} \quad \sigma = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 4 & 5 & 6 & \circ 7 & \circ 14 \\ \hline 2 & \circ 8 & \circ 13 & & & \\ \hline 3 & \circ 9 & & & & \\ \hline \circ 10 & \circ 12 & & & & \\ \hline \circ 11 & & & & & \\ \hline \end{array}$$

From Proposition 4.3 we obtain the following theorem.

THEOREM 1.1

Fix A and its divisions (U, C) and (A, B) . Fix another A' and its divisions (U', C') and (A', B') . The map in Definition 4.9 from admissible matrices $a \in \mathcal{M}(A', B', A, B)$ to pairs (π, σ) , where π is (A, B) -partially strict tableau, σ is (A', B') -partially strict tableau and π and σ have the same shape, is a bijection.

The following proposition is also easy to see from definitions.

PROPOSITION 1.4

Let $a \in \mathcal{M}(A', B', A, B)$. If $p(a)$ correspond to a by the map in Definition 4.10, then the inverse biword $p(a)^{-1}$ correspond to a^t . Here a^t denote the conjugate matrix of a .

From Proposition 4.4 we obtain the following theorem.

THEOREM 1.2

Fix A and its divisions (U, C) and (A, B) . Assume that (π, σ) correspond to a by the bijection in Definition 4.9, where $a \in \mathcal{M}(A, B, A, B)$, and π and σ are (A, B) -partially strict tableau having the same shape. Then (σ, π) correspond to a^t by the same bijection.

EXAMPLE 1.12

Let a be as Example 4.7. Then

$$a^t = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 \end{pmatrix}$$

and

$$l(a) = \begin{pmatrix} 1 & 1 & 1 & \circ 2 & \circ 2 & \circ 2 & \circ 2 & \circ 3 & \circ 3 & \circ 3 & 4 & 4 & 4 & 4 \\ 1 & 1 & \circ 4 & \circ 3 & \circ 3 & \circ 3 & 2 & 2 & \circ 3 & \circ 4 & \circ 4 & \circ 4 & 2 & 2 \end{pmatrix}.$$

It's easy to make sure that

$$\pi = \begin{array}{|c|c|c|c|c|c|} \hline 1 & \circ 2 & \circ 3 & \circ 3 & 4 & 4 \\ \hline 1 & \circ 2 & 4 & & & \\ \hline 1 & \circ 3 & & & & \\ \hline \circ 2 & 4 & & & & \\ \hline \circ 2 & & & & & \\ \hline \end{array} \quad \sigma = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 2 & 2 & \circ 3 & \circ 4 \\ \hline 1 & \circ 3 & \circ 4 & & & \\ \hline 2 & \circ 3 & & & & \\ \hline \circ 3 & \circ 4 & & & & \\ \hline \circ 4 & & & & & \\ \hline \end{array}$$

DEFINITION 1.11

Fix A and its divisions (U, C) and (A, B) . Let $a = (a_{ij})_{i,j \in A} \in \mathcal{M}(A, B, A, B)$ be an admissible symmetric matrix. We define $\text{tr}_{(A,B)} a$ by

$$\text{tr}_{(A,B)} a = \sum_{i \in A} a_{ii} + \sum_{i \in B} \text{odd}\{a_{ii}\}$$

where $\text{odd}\{x\} = \begin{cases} 1 & \text{if } x \text{ is odd} \\ 0 & \text{if } x \text{ is even} \end{cases}$.

COROLLARY 1.1

Fix A and its divisions (U, C) and (A, B) . The map in Definition 4.9 gives a bijection from admissible symmetric matrices $a \in \mathcal{M}(A, B, A, B)$ onto (A, B) -partially strict tableaux π . In this bijection we have

$$\text{tr}_{(A,B)} a = \text{odd}(\lambda)$$

where λ is the shape of π and $\text{odd}(\lambda)$ stands for the number of odd length columns in λ .

EXAMPLE 1.13

Let $A = \{1, \circ 3\}$ and $B = \{\circ 2, 4\}$. Let a be an admissible symmetric matrix given by

$$a = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then

$$l(a) = \begin{pmatrix} 1 & 1 & 1 & \circ 2 & \circ 2 & \circ 2 & \circ 3 & \circ 3 & \circ 3 & \circ 3 & 4 & 4 \\ \circ 3 & \circ 3 & 1 & \circ 3 & \circ 2 & \circ 2 & 4 & 1 & 1 & \circ 2 & 4 & \circ 3 \end{pmatrix}.$$

and

$$\pi = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & \circ 2 & 4 \\ \hline \circ 2 & \circ 3 & \circ 3 & \circ 3 & \\ \hline \circ 2 & 4 & & & \\ \hline \circ 3 & & & & \\ \hline \end{array}$$

COROLLARY 1.2

Fix A and its division (A, B) .

$$\sum_{\lambda} HS_{\lambda}^{(A,B)}(x) t^{\text{odd}(\lambda)} = \prod_{\substack{(i,j) \in A \times A \cup B \times B \\ i < j}} \frac{1}{1 - x_i x_j} \prod_{(i,j) \in A \times B} (1 + x_i x_j) \prod_{i \in A} \frac{1}{1 - t x_i} \prod_{i \in B} \frac{1 + t x_i}{1 - x_i^2}$$

In particular,

$$\sum_{\lambda' \text{ even}} HS_{\lambda}^{(A,B)}(x) = \prod_{\substack{(i,j) \in A \times A \cup B \times B \\ i < j}} \frac{1}{1 - x_i x_j} \prod_{(i,j) \in A \times B} (1 + x_i x_j) \prod_{i \in B} \frac{1}{1 - x_i^2}$$

Now we investigate skew case. Let $\text{PST}_{(A,B)}(\lambda/\mu)$ denote the set of (A, B) -partially strict skew tableaux which have skew shape λ/μ .

THEOREM 1.3

Fix A and its divisions (U, C) and (A, B) . Fix another A' and its divisions (U', C') and (A', B') . Let α and β be fixed partitions. Then the map

$$(a, \tau, \kappa) \mapsto (\pi, \sigma)$$

defined below is a bijection between admissible matrices $a \in \mathcal{M}(A, B, A', B')$ with $\tau \in \text{PST}_{(A,B)}(\alpha/\mu)$ and $\kappa \in \text{PST}_{(A',B')}(\beta/\mu)$, on the one hand, and $\pi \in \text{PST}_{(A,B)}(\lambda/\beta)$ and $\sigma \in \text{PST}_{(A',B')}(\lambda/\alpha)$, on the other, such that $\tau \cup \hat{l}(a) = \pi$ and $\kappa \cup \hat{l}(a) = \sigma$.

Proof.

Let the largest letter of $\kappa \cup \widehat{l}(a)$ be n . We construct (π_r, σ_r) for $r = 0, 1, \dots, n$ as follows. Start with $(\pi_0, \sigma_0) = (\tau, \emptyset_\alpha)$. Form π_r from π_{r-1} as follows.

Case 1 : $r \in A'_u \cup B'_c$

At first we insert all the circled letters of $\widehat{l}(a)$ paired with r 's in $\widehat{l}(a)$, where these circled letters are arranged in decreasing order. Next we internally insert all the letters of π_{r-1} corresponding to r 's in σ_{r-1} . If $r \in A'_u$, the insertion proceed left to right, and if $r \in B'_c$, the insertion proceed top to bottom. Finally we insert all the uncircled letters of $\widehat{l}(a)$ paired with r 's in $\widehat{l}(a)$, where these uncircled letters are arranged in increasing order.

Case 2 : $r \in A'_c \cup B'_u$

At first we insert all the uncircled letters of $\widehat{l}(a)$ paired with r 's in $\widehat{l}(a)$, where these uncircled letters are arranged in decreasing order. Next we internally insert all the letters of π_{r-1} corresponding to r 's in σ_{r-1} . If $r \in A'_c$, the insertion proceed left to right, and if $r \in B'_u$, the insertion proceed top to bottom. Finally we insert all the circled letters of $\widehat{l}(a)$ paired with r 's in $\widehat{l}(a)$, where these uncircled letters are arranged in increasing order.

In either case placing r 's in the appropriate cells of σ_{r-1} result in σ_r . It is not hard to see that the cells where r 's are placed are horizontal or vertical strip in σ_r . At last we put $(\pi_n, \sigma_n) = (\pi, \sigma)$. ■

EXAMPLE 1.14

Let $A = \{1, \circ 2\}$, $B = \{\circ 3, 4\}$, $A' = \{1, \circ 3\}$, $B' = \{2, \circ 4\}$, $\alpha = (221)$ and $\beta = (43)$. Let $a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

so that the matrix word of a is $l(a) = \begin{pmatrix} 1 & 2 & 2 & \circ 3 & \circ 3 & \circ 3 & \circ 4 \\ \circ 2 & 4 & \circ 2 & 1 & 1 & \circ 2 & \circ 3 \end{pmatrix}$. Let

$$\tau = \begin{array}{|c|c|} \hline & 1 \\ \hline 1 & 4 \\ \hline \circ 3 & \\ \hline \end{array} \quad \kappa = \begin{array}{|c|c|c|c|} \hline & 1 & 2 & \circ 4 \\ \hline 1 & \circ 3 & \circ 4 & \\ \hline \end{array}$$

Then we have

$$\pi = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & 1 & 1 & \circ 3 & \circ 4 \\ \hline & & & 1 & \circ 2 & & & \\ \hline 1 & \circ 2 & 4 & & & & & \\ \hline \circ 2 & \circ 3 & & & & & & \\ \hline \end{array} \quad \sigma = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & 1 & 1 & 2 & \circ 3 & \circ 3 & \circ 4 \\ \hline & & 2 & \circ 3 & \circ 3 & & & \\ \hline & 1 & \circ 4 & & & & & \\ \hline 2 & \circ 4 & & & & & & \\ \hline \end{array}$$

COROLLARY 1.3

Fix A and its division (A, B) . Fix another A' and its division (A', B') . Let α and β be fixed partitions. Then

$$\begin{aligned} & \sum_{\lambda} HS_{\lambda/\beta}^{(A,B)}(x) HS_{\lambda/\alpha}^{(A',B')}(y) \\ &= \sum_{\mu} HS_{\alpha/\mu}^{(A,B)}(x) HS_{\beta/\mu}^{(A',B')}(y) \prod_{(i,j) \in A \times A' \cup B \times B'} \frac{1}{1 - x_i y_j} \prod_{(i,j) \in A \times B' \cup B \times A'} (1 + x_i y_j) \end{aligned}$$

THEOREM 1.4

Let $A = A'$, $(A, B) = (A', B')$, $(U, C) = (U', C')$ and $\alpha = \beta$ in Theorem 4.3. If (a, τ, κ) correspond to (π, σ) by the bijection in Theorem 4.3 then (a^t, κ, τ) correspond to (σ, π) by the same bijection.

THEOREM 1.5

Fix A and its divisions (U, C) and (A, B) . Let α be a fixed partition. Then the mapping in Theorem 4.3 restricts to a bijection

$$(a, \tau) \leftrightarrow \pi$$

where $a \in \mathcal{M}(A, B, A, B)$ is a symmetric matrix, $\tau \in \text{PST}_{(A, B)}(\alpha/\mu)$, $\pi \in \text{PST}_{(A, B)}(\lambda/\mu)$, and $\bar{l}(a) \cup \tau = \pi$. In this bijection we always have

$$\text{tr}_{(A, B)} a + \text{odd}(\mu) = \text{odd}(\lambda)$$

EXAMPLE 1.15

Let $A = \{1, {}^\circ 3\}$, $B = \{{}^\circ 2, 4\}$ and $\alpha = (221)$. Let $a = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ so that the matrix word of a is

$$l(a) = \begin{pmatrix} 1 & 1 & 1 & {}^\circ 2 & {}^\circ 2 & {}^\circ 2 & {}^\circ 3 & 4 \\ {}^\circ 3 & {}^\circ 2 & 4 & {}^\circ 2 & {}^\circ 2 & 1 & 1 & 1 \end{pmatrix}. \text{ Let}$$

$$\tau = \begin{array}{|c|c|} \hline & \\ \hline & 4 \\ \hline {}^\circ 3 & \\ \hline \end{array}$$

Then we have

$$\pi = \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline & & {}^\circ 2 & {}^\circ 3 \\ \hline & 1 & {}^\circ 2 & 4 \\ \hline {}^\circ 2 & {}^\circ 3 & 4 & \\ \hline \end{array}$$

COROLLARY 1.4

Fix A and its division (A, B) . Let α be a fixed partition.

$$\begin{aligned} & \sum_{\lambda} HS_{\lambda/\alpha}^{(A, B)}(x) t^{\text{odd}(\lambda)} \\ &= \sum_{\mu} HS_{\alpha/\mu}^{(A, B)}(x) t^{\text{odd}(\mu)} \prod_{\substack{(i, j) \in A \times A \cup B \times B \\ i < j}} \frac{1}{1 - x_i x_j} \prod_{(i, j) \in A \times B} (1 + x_i x_j) \prod_{i \in A} \frac{1}{1 - t x_i} \prod_{i \in B} \frac{1 + t x_i}{1 - x_i^2} \end{aligned}$$

In particular,

$$\sum_{\lambda' \text{ even}} HS_{\lambda'/\alpha}^{(A, B)}(x) = \sum_{\mu' \text{ even}} HS_{\alpha/\mu'}^{(A, B)}(x) \prod_{\substack{(i, j) \in A \times A \cup B \times B \\ i < j}} \frac{1}{1 - x_i x_j} \prod_{(i, j) \in A \times B} (1 + x_i x_j) \prod_{i \in B} \frac{1}{1 - x_i^2}$$

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